Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

AMATH 731 Final Project

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Outline

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- Want to know what restrictions there are on f and T

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Hilbert Space & Fourier Series

- Recall for any separable Hilbert space H
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- This is the Fourier series of x in (x_k)

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For any $f \in L_2([-b, b])$,

$$f(t) = \sum_k \frac{1}{\sqrt{2b}} \hat{f}(k) e^{i\pi kt/b}$$

where

$$\hat{f}(k) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(t) e^{-i\pi kt/b} dt.$$

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- Call F the Fourier Transform of f

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- Can extend domain to include $L_2(\mathbb{R})$ and even infinitely periodic function

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- In applications, x will be related to the frequencies of what a signal f represents
- Ex. If f(t) is a sound wave, and humans can only hear up to 22kHz, we often set frequencies above 22kHz to zero since we can't hear them anyway

Fourier Transform Inverse

Fourier Transform Inversion Theorem

Let $f \in L_1(\mathbb{R})$, $F = \mathcal{F}f$. Define $g \in C(\mathbb{R})$ as

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)e^{ixt}dx.$$

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- g is continuous, so equality on $\mathbb{R}$ if f is continuous

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- Show $\lim_{\lambda\to 0^+}(f * h_\lambda) = f$ (in L_1 norm)

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Let $f \in L_1(\mathbb{R})$, continuous, and Ω-bandlimited. And suppose $\mathcal{F}f = F \in L_2(\mathbb{R})$.
$\implies$ We can completely reconstruct $f(t)$ (pointwise) using only samples $f_k = f(kT)$ where $T = \pi/\Omega$.

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$\implies$ We can completely reconstruct $f(t)$ (pointwise) using only samples $f_k = f(kT)$ where $T = \pi/\Omega$.

- Can instead force continuity and piecewise smoothness on F for uniform convergence.
- Can actually use any $T \in (0, \pi/\Omega)$

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- $f(-k\pi/\Omega)$ completely determines $\hat{F}(k)$, F, and hence $f(t)$.

Whittaker–Shannon Interpolation

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- How do we actually recover f?
- Plug in these three formulas and simplify

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ixt} dx \tag{1}$$

$$F(x) = \sum_{k} \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega} \tag{2}$$

$$\hat{F}(k) = \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} f(-k\pi/\Omega) \tag{3}$$

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- (2) into (1)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \left[\sum_k \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega} \right] e^{ixt} dx$$

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$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \sum_k \left(\hat{F}(k) \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx \right)$$

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- (3) into $\hat{F}(k)$

$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \sum_k \left[\frac{\sqrt{2\pi}}{\sqrt{2\Omega}} f(-k\pi/\Omega) \right] \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx$$

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$$f(t) = \frac{1}{2\Omega} \sum_k \left(f(-k\pi/\Omega) \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx \right)$$

- Evaluate Integral

$$\int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega+t)} dx = \frac{1}{i(\pi k/\Omega + t)} \left[e^{ix(\pi k/\Omega+t)} \right]_{x=-\Omega}^{x=\Omega}$$
$$= \frac{1}{i(\pi k/\Omega + t)} \left[e^{i(\pi k+\Omega t)} - e^{-i(\pi k+\Omega t)} \right]$$
$$= \frac{2\Omega}{\pi k + \Omega t} \sin(\pi k + \Omega t)$$

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Where

$$\operatorname{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{sin(\pi x)}{\pi x} & x \neq 0 \end{cases}$$

Whittaker–Shannon Interpolation

Summary

Whittaker–Shannon Interpolation

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- Ω-bandlimited implies Ω_2-bandlimited for any $\Omega_2 > \Omega$

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- CDs have a sampling rate of 44.1kHz since humans can only hear up to around 22kHz
- Often apply a “lowpass filter” in applications before sampling to ensure functions are bandlimited

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References

A. Boggess and F.J. Narcowich. *A First Course in Wavelets with Fourier Analysis*. Wiley, 2011. ISBN: 9781118211151.

Tomasz Kochanek. *Convolution and Fourier transform; Fourier inversion formula and the uniqueness theorem for Fourier transform*. 2018. URL: impan.pl/~tkoch/FA_lecturenotes/lecture14.pdf.

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  Ex: Non-zero constant functions are in L_∞ but not the other two