# Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation AMATH 731 Final Project

Nicholas Richardson

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Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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## Goal

▶ Say we have a function  $f : \mathbb{R} \to \mathbb{R}$ 

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### Goal

- ▶ Say we have a function  $f : \mathbb{R} \to \mathbb{R}$
- ▶ Want to recreate f(t) using only samples  $f_k = f(kT)$ ,  $k \in \mathbb{Z}$ , (0 < T fixed)

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- ▶ Say we have a function  $f: \mathbb{R} \to \mathbb{R}$
- ▶ Want to recreate f(t) using only samples  $f_k = f(kT)$ ,  $k \in \mathbb{Z}$ , (0 < T fixed)
- ▶ Want formula  $f(t) = S(\{f_k\}, t)$

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### Goal

- ▶ Say we have a function  $f : \mathbb{R} \to \mathbb{R}$
- ▶ Want to recreate f(t) using only samples  $f_k = f(kT)$ ,  $k \in \mathbb{Z}$ , (0 < T fixed)
- ▶ Want formula  $f(t) = S(\{f_k\}, t)$
- ▶ Want to know what restrictions there are on f and T

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# Hilbert Space & Fourier Series

- ▶ Recall for any separable Hilbert space *H*
- $\triangleright$   $(x_k)$  orthonormal basis

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# Hilbert Space & Fourier Series

- ► Recall for any separable Hilbert space *H*
- $\triangleright$   $(x_k)$  orthonormal basis

$$x = \sum_{k} \langle x, x_k \rangle x_k, \forall x \in H$$

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# Hilbert Space & Fourier Series

- ▶ Recall for any separable Hilbert space *H*
- $\triangleright$   $(x_k)$  orthonormal basis

$$x = \sum_{k} \langle x, x_k \rangle x_k, \forall x \in H$$

▶ This is the Fourier series of x in  $(x_k)$ 

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▶ Consider  $H = L_2([-b, b])$  ,  $0 < b < \infty$ 

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#### Application

- ► Consider  $H = L_2([-b, b])$  ,  $0 < b < \infty$
- ightharpoonup Will allow  $f: \mathbb{R} \to \mathbb{C}$

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- ► Consider  $H = L_2([-b, b])$ ,  $0 < b < \infty$
- ightharpoonup Will allow  $f: \mathbb{R} \to \mathbb{C}$
- $u_k(t) = \frac{1}{\sqrt{2b}} e^{i\pi kt/b}$  forms an orthonormal basis

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Application

- ► Consider  $H = L_2([-b, b])$  ,  $0 < b < \infty$
- ightharpoonup Will allow  $f: \mathbb{R} \to \mathbb{C}$
- $\mathbf{v}_k(t) = \frac{1}{\sqrt{2b}} e^{i\pi kt/b}$  forms an orthonormal basis

### Classical Fourier Series

For any  $f \in L_2([-b, b])$ ,

$$f(t) = \sum_{k} \frac{1}{\sqrt{2b}} \hat{f}(k) e^{i\pi kt/b}$$

where

$$\hat{f}(k) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(t) e^{-i\pi kt/b} dt.$$

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### Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(t) e^{-i\pi kt/b} dt.$$

▶ "Take  $b \to \infty$ "

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### Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(t) e^{-i\pi kt/b} dt.$$

• "Take  $b \to \infty$ "

### (Forward) Fourier Transform

Define  $\mathcal{F}: L_1(\mathbb{R}) \to L_\infty(\mathbb{R})$  as  $\mathcal{F}f = F$  where

$$F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt.$$

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### Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(t) e^{-i\pi kt/b} dt.$$

• "Take  $b \to \infty$ "

### (Forward) Fourier Transform

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$$F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt.$$

Call F the Fourier Transform of f

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### Fourier Transform Notes

▶ Use  $L_1(\mathbb{R})$  to ensure integral exists

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### Fourier Transform Notes

▶ Use  $L_1(\mathbb{R})$  to ensure integral exists

$$||F||_{\infty} \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| ||e^{-ixt}||_{\infty} dt = \frac{1}{\sqrt{2\pi}} ||f||_{1}$$

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### Fourier Transform Notes

▶ Use  $L_1(\mathbb{R})$  to ensure integral exists

$$\|F\|_{\infty} \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| \left\| e^{-ixt} \right\|_{\infty} dt = \frac{1}{\sqrt{2\pi}} \left\| f \right\|_{1}$$

▶ Can extend domain to include  $L_2(\mathbb{R})$  and even infinitely periodic function

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### **Bandlimited Functions**

• f is Ω-bandlimited if F(x) = 0 for |x| > Ω

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### **Bandlimited Functions**

- f is Ω-bandlimited if F(x) = 0 for  $|x| > \Omega$
- ► In applications, x will be related to the frequencies of what a signal f represents

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### **Bandlimited Functions**

- f is Ω-bandlimited if F(x) = 0 for  $|x| > \Omega$
- ► In applications, *x* will be related to the frequencies of what a signal *f* represents
- ► Ex. If f(t) is a sound wave, and humans can only hear up to 22kHz, we often set frequencies above 22kHz to zero since we can't hear them anyway

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### Fourier Transform Inversion Theorem

Let  $f \in L_1(\mathbb{R})$ ,  $F = \mathcal{F}f$ . Define  $g \in C(\mathbb{R})$  as

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ixt} dx.$$

Then g = f a.e.

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$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ixt} dx.$$

Then g = f a.e.

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**Proof Sketch** 

- Can't plug in forward transform (non convergent integral)
- ▶ Define  $h_{\lambda}(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2}$  (Dirac Delta Approximator)

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- Can't plug in forward transform (non convergent integral)
- ▶ Define  $h_{\lambda}(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2}$  (Dirac Delta Approximator)
- ► Define convolution:

$$(f*h_{\lambda})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)h_{\lambda}(y)dy.$$

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### Can't plug in forward transform (non convergent integral)

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- Define convolution:

$$(f*h_{\lambda})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)h_{\lambda}(y)dy.$$

▶ Show  $\lim_{\lambda \to 0^+} (f * h_{\lambda})(x) = g(x)$  (pointwise)

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## Can't plug in forward transform (non convergent integral)

- ▶ Define  $h_{\lambda}(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2}$  (Dirac Delta Approximator)
- Define convolution:

$$(f*h_{\lambda})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)h_{\lambda}(y)dy.$$

- ▶ Show  $\lim_{\lambda \to 0^+} (f * h_{\lambda})(x) = g(x)$  (pointwise)
- ▶ Show  $\lim_{\lambda \to 0^+} (f * h_{\lambda}) = f$  (in  $L_1$  norm)

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# Nyquist-Shannon Theorem

# Nyquist-Shannon Sampling Theorem

Let  $f \in L_1(\mathbb{R})$ , continuous, and  $\Omega$ -bandlimited. And suppose  $\mathcal{F}f = F \in L_2(\mathbb{R})$ .

 $\implies$  We can completely reconstruct f(t) (pointwise) using only samples  $f_k = f(kT)$  where  $T = \pi/\Omega$ .

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# Nyquist-Shannon Theorem

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### Nyquist-Shannon Sampling Theorem

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 $\implies$  We can completely reconstruct f(t) (pointwise) using only samples  $f_k = f(kT)$  where  $T = \pi/\Omega$ .

► Can instead force continuity and piecewise smoothness on *F* for uniform convergence.

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### Nyquist-Shannon Sampling Theorem

Let  $f \in L_1(\mathbb{R})$ , continuous, and  $\Omega$ -bandlimited. And suppose  $\mathcal{F}f = F \in L_2(\mathbb{R}).$ 

 $\implies$  We can completely reconstruct f(t) (pointwise) using only samples  $f_k = f(kT)$  where  $T = \pi/\Omega$ .

- Can instead force continuity and piecewise smoothness on F for uniform convergence.
- ightharpoonup Can actually use any  $T \in (0, \pi/\Omega)$

Nyquist-Shannon Theorem

▶ Write the Fourier series of *F* 

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### Nyquist-Shannon Theorem

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### Application

▶ Write the Fourier series of *F* 

$$F(x) = \sum_{k} \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega},$$

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Proof

▶ Write the Fourier series of *F* 

$$F(x) = \sum_{k} \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega},$$

where

$$\hat{F}(k) = \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} F(x) e^{-i\pi kx/\Omega} dx.$$

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lacktriangle Extend integration to all of  $\mathbb R$  and rearrange constants

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## Nyquist-Shannon Theorem

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ightharpoonup Extend integration to all of  $\mathbb R$  and rearrange constants

$$\hat{F}(k) = \frac{1}{\sqrt{2\Omega}} \int_{-\infty}^{\infty} F(x) e^{-i\pi kx/\Omega} dx$$

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## Nyquist-Shannon Theorem

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ightharpoonup Extend integration to all of  $\mathbb R$  and rearrange constants

$$\hat{F}(k) = \frac{1}{\sqrt{2\Omega}} \int_{-\infty}^{\infty} F(x) e^{-i\pi kx/\Omega} dx$$
$$= \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ix(-k\pi/\Omega)} dx$$

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$$\hat{F}(k) = \frac{1}{\sqrt{2\Omega}} \int_{-\infty}^{\infty} F(x) e^{-i\pi kx/\Omega} dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ix(-k\pi/\Omega)} dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} f(-k\pi/\Omega).$$

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•  $f(-k\pi/\Omega)$  completely determines  $\hat{F}(k)$ , F, and hence f(t).

Extend integration to all of  $\mathbb{R}$  and rearrange constants

 $\hat{F}(k) = \frac{1}{\sqrt{2\Omega}} \int_{-\infty}^{\infty} F(x) e^{-i\pi kx/\Omega} dx$ 

 $=\frac{\sqrt{2\pi}}{\sqrt{2\Omega}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{ix(-k\pi/\Omega)}dx$ 

 $=\frac{\sqrt{2\pi}}{\sqrt{2\Omega}}f(-k\pi/\Omega).$ 

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► How do we actually recover *f*?

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- ► How do we actually recover *f*?
- ▶ Plug in these three formulas and simplify

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)e^{ixt} dx \tag{1}$$

$$F(x) = \sum_{k} \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega}$$
 (2)

$$\hat{F}(k) = \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} f(-k\pi/\Omega) \tag{3}$$

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Proof

▶ (2) into (1)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \left[ \sum_{k} \frac{1}{\sqrt{2\Omega}} \hat{F}(k) e^{i\pi kx/\Omega} \right] e^{ixt} dx$$

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Proof

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$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \sum_{k} \left( \hat{F}(k) \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx \right)$$

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 $\blacktriangleright$  (3) into  $\hat{F}(k)$ 

$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \sum_{k} \left[ \frac{\sqrt{2\pi}}{\sqrt{2\Omega}} f(-k\pi/\Omega) \right] \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx$$

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$$f(t) = \frac{1}{2\Omega} \sum_{k} \left( f(-k\pi/\Omega) \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx \right)$$

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Application

$$f(t) = \frac{1}{2\Omega} \sum_{k} \left( f(-k\pi/\Omega) \int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx \right)$$

Evaluate Integral

$$\int_{-\Omega}^{\Omega} e^{ix(\pi k/\Omega + t)} dx = \frac{1}{i(\pi k/\Omega + t)} \left[ e^{ix(\pi k/\Omega + t)} \right]_{x=-\Omega}^{x=\Omega}$$

$$= \frac{1}{i(\pi k/\Omega + t)} \left[ e^{i(\pi k+\Omega t)} - e^{-i(\pi k+\Omega t)} \right]$$

$$= \frac{2\Omega}{\pi k + \Omega t} \sin(\pi k + \Omega t)$$

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Application

▶ Plug back into formula for f(t)

$$f(t) = \frac{1}{2\Omega} \sum_{k} \left( f(-k\pi/\Omega) \frac{2\Omega}{\pi k + \Omega t} \sin(\pi k + \Omega t) \right)$$

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$$f(t) = rac{1}{2\Omega} \sum_{k} \left( f(-k\pi/\Omega) rac{2\Omega}{\pi k + \Omega t} \sin(\pi k + \Omega t) 
ight)$$

$$= \sum_{k} f(-k\pi/\Omega) rac{\sin(\pi k + \Omega t)}{\pi k + \Omega t}$$

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# Main I

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Application

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$$f(t) = \frac{1}{2\Omega} \sum_{k} \left( f(-k\pi/\Omega) \frac{2\Omega}{\pi k + \Omega t} \sin(\pi k + \Omega t) \right)$$
$$= \sum_{k} f(-k\pi/\Omega) \frac{\sin(\pi k + \Omega t)}{\pi k + \Omega t}$$
$$= \sum_{k} f\left(\frac{k\pi}{\Omega}\right) \operatorname{sinc}\left(\frac{\Omega t}{\pi} - k\right)$$

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$$= \sum_{k} f\left(\frac{k\pi}{\Omega}\right) \operatorname{sinc}\left(\frac{\Omega t}{\pi} - k\right)$$

Where

$$\operatorname{sinc}(x) = \begin{cases} 1 & x = 0\\ \frac{\sin(\pi x)}{\pi x} & x \neq 0 \end{cases}$$

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Summary

# Whittaker-Shannon Interpolation

Given the Nyquist-Shannon Sampling Theorem assumptions hold,

$$f(t) = \sum_{k} f_{k} \operatorname{sinc}\left(\frac{t}{T} - k\right)$$

where  $f_k = f(kT)$ ,  $T = \pi/\Omega$ .

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Summary

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 $\nu = \frac{\Omega}{2\pi}$ Hz Nyquist Frequency

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Summary

# Whittaker–Shannon Interpolation

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where  $f_k = f(kT)$ ,  $T = \pi/\Omega$ .

- $\nu = \frac{\Omega}{2\pi}$ Hz Nyquist Frequency
- $\nu_s = \frac{1}{T} = 2\nu \text{Hz Nyquist Rate}$

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Given the Nyquist-Shannon Sampling Theorem assumptions hold.

$$f(t) = \sum_{k} f_{k} \operatorname{sinc}\left(\frac{t}{T} - k\right)$$

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- $\nu = \frac{\Omega}{2\pi}$ Hz Nyquist Frequency
- $\nu_s = \frac{1}{T} = 2\nu \text{Hz Nyquist Rate}$
- Can and should always sample faster (smaller T or bigger  $\nu_s$ )

Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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Whittaker-Shannon Interpolation

Given the Nyquist-Shannon Sampling Theorem assumptions hold,

$$f(t) = \sum_{k} f_{k} \operatorname{sinc}\left(\frac{t}{T} - k\right)$$

where  $f_k = f(kT)$ ,  $T = \pi/\Omega$ .

- $\nu = \frac{\Omega}{2\pi}$ Hz Nyquist Frequency
- $\nu_s = \frac{1}{T} = 2\nu Hz$  Nyquist Rate
- ► Can and should always sample faster (smaller T or bigger  $\nu_s$ )
- ightharpoonup Ω-bandlimited implies  $\Omega_2$ -bandlimited for any  $\Omega_2 > \Omega$

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# **Application**

► Most human voice frequencies are below 3.5kHz, so telephones sample at 8kHz.

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# **Application**

- ► Most human voice frequencies are below 3.5kHz, so telephones sample at 8kHz.
- ► CDs have a sampling rate of 44.1kHz since humans can only hear up to around 22kHz

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# **Application**

- ► Most human voice frequencies are below 3.5kHz, so telephones sample at 8kHz.
- ► CDs have a sampling rate of 44.1kHz since humans can only hear up to around 22kHz
- Often apply a "lowpass filter" in applications before sampling to ensure functions are bandlimited

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▶ For any bounded interval I,  $L_{\infty}(I) \subset L_2(I) \subset L_1(I)$ 

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## pplication

For any bounded interval I,  $L_{\infty}(I) \subset L_2(I) \subset L_1(I)$ Ex:  $1/\sqrt{x}$  is in  $L_1([0,1])$  but not in the other two Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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Application

For any bounded interval I,  $L_{\infty}(I) \subset L_2(I) \subset L_1(I)$ Ex:  $1/\sqrt{x}$  is in  $L_1([0,1])$  but not in the other two Ex:  $1/x^{1/4}$  is in  $L_2([0,1])$  but not in  $L_{\infty}([0,1])$  Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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Application

For any bounded interval I,  $L_{\infty}(I) \subset L_2(I) \subset L_1(I)$ Ex:  $1/\sqrt{x}$  is in  $L_1([0,1])$  but not in the other two Ex:  $1/x^{1/4}$  is in  $L_2([0,1])$  but not in  $L_{\infty}([0,1])$ 

▶ But  $L_{\infty}(\mathbb{R}) \not\subset L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$ 

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Application

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▶ But  $L_{\infty}(\mathbb{R}) \not\subset L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$ Ex: 1/x for  $x \geq 1$ , 0 else, is in  $L_2(\mathbb{R})$  but not  $L_1(\mathbb{R})$  Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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But  $L_{\infty}(\mathbb{R}) \not\subset L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$ Ex: 1/x for  $x \geq 1$ , 0 else, is in  $L_2(\mathbb{R})$  but not  $L_1(\mathbb{R})$ Ex: Non-zero constant functions are in  $L_{\infty}$  but not the other two Nyquist-Shannon Sampling Theorem and Whittaker-Shannon Interpolation

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